

STEP MATHEMATICS 2

2019

Mark Scheme

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1		$f'(x) = g(x) + (x - p)g'(x)$ Tangent passes through $(a, (a - p)g(a))$ Equation of tangent is $y = (g(a) + (a - p)g'(a))(x - a) + (a - p)g(a)$ (or equivalent equation) Substitution of $x = p$ into equation of tangent $y = -(a - p)^2g'(a)$ Verification that if $g'(a) = 0$, then $y = 0$ If $y = 0$ then $g'(a) = 0$ because $a \neq p$	M1 M1 A1 E1 E1 E1 (AG) (6 marks)
	(i)	$g(x) = A(x - q)(x - r)$ identified $g'(a) = 0 \Rightarrow 2a = r + q$ (legitimately obtained) Gradient of tangent is $g(a) + (a - p)g'(a)$ $= A(a - q)(a - r)$ $= -\frac{1}{4}A(r - q)^2$	M1 A1 (AG) M1 A1 (4 marks)
	(ii)	By symmetry, the gradient of the second tangent is $-\frac{1}{4}A(p - q)^2$ (can be implied) Parallel iff $(p - q)^2 = (q - r)^2$ $\Leftrightarrow q - p = r - q$ since $p < q < r$. Tangent at $x = q$, $y = A(q - p)(q - r)(x - q)$, Meets curve again when $(q - p)(q - r)(x - q) = (x - p)(x - r)(x - q)$ $\Leftrightarrow (q - p)(q - r) = (x - p)(x - r)$ since $x \neq q$ (cancellation must be justified for M1, can be awarded if used correctly on $(x - q)^2(x - p - r + q)$ later) $\Leftrightarrow (x - q)(x - p - r + q) = 0$ $\Leftrightarrow x = p + r - q$ or $x = q$ Therefore there is only one point of intersection between the tangent and the curve if and only if $p + r - q = q$, which is if and only if the tangents are parallel. One E mark for each direction.	B1 M1 A1 E1 M1 M1 M1 A1 E1 E1 (AG) (10 marks)

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2		<p>Sketch with areas $\int_0^x f(t) dt$, $\int_0^{f(x)} f^{-1}(y) dy$ and rectangle correctly identified. (One mark any one)</p>	<p>G1 G1 (2 marks)</p>
	(i)	<p>$g(0)(g(0)^2 + 1) = 0$ factorised $g(0)$ real so $g(0) = 0$ (must be justified)</p> <p>$1 = (3g(t)^2 + 1)g'(t)$ $(3g(t)^2 + 1) > 0$ so $g'(t) > 0$</p> <p>$g(2)^3 + g(2) - 2 = 0$ $(g(2) - 1)(g(2)^2 + g(2) + 2) = 0$ $\Delta = -7 < 0$ so $g(2) = 1$ or $g(2) > 0$ justified</p> <p>$g^{-1}(s) = s^3 + s$ $\int_0^2 g(t) dt = 2g(2) - \int_0^{g(2)} g^{-1}(s) ds$ $= \frac{5}{4}$</p>	<p>M1 A1 (AG)</p> <p>M1 A1 (AG)</p> <p>M1 A1</p> <p>B1 M1 A1</p> <p>(9 marks)</p>
	(ii)	<p>$h(t) = g(t + 2)$ so $h(0) = g(2) = 1$ and $h'(t) > 0$</p> <p>$(h(8) - 2)(h(8)^2 + 2h(8) + 5) = 0$ $h(8) = 2$ correctly justified</p> <p>$h^{-1}(s) = s^3 + s - 2$ $\int_0^8 h(t) dt + \int_{h(0)}^{h(8)} h^{-1}(s) ds = 16$ (or similar correct equation) $\int_0^8 h(t) dt = 16 - \int_1^2 (s^3 + s - 2) ds$ $= 16 - [\frac{s^4}{4} + \frac{s^2}{2} - 2s]_1^2$ (integration) $= 12\frac{3}{4}$</p>	<p>M1 A1</p> <p>M1 A1</p> <p>B1 M1 A1</p> <p>M1 A1</p> <p>(9 marks)</p>

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3		<p>$x_1 + x_2$ is maximised when both have the same sign, In which case $x_1 + x_2 = x_1 + x_2$. Thus, $x_1 + x_2 \leq x_1 + x_2$ (or by consideration of all four combinations of signs separately)</p> <p>$x_1 + \dots + x_{n-1} + x_n \leq x_1 + \dots + x_{n-1} + x_n$ $\leq \dots$ $\leq x_1 + \dots + x_{n-1} + x_n$ by induction</p>	<p>E1</p> <p>E1</p> <p>(2 marks)</p>
	(i) (a)	<p>$f(x) - 1 = a_1x + \dots + a_{n-1}x^{n-1} + x^n$ $\leq a_1x + \dots + a_{n-1}x^{n-1} + x^n$ $= a_1 x + \dots + a_{n-1} x ^{n-1} + x ^n$ $\leq A(x + \dots + x ^{n-1}) + x ^n$ $\leq A(x + \dots + x ^{n-1} + x ^n)$ (justified) $= A \frac{ x (1- x ^n)}{1- x }$ $\leq A \frac{ x }{1- x }$ (justified)</p>	<p>M1</p> <p>M1</p> <p>M1</p> <p>M1</p> <p>M1</p> <p>A1 (AG)</p> <p>(6 marks)</p>
	(b)	<p>$1 \leq \frac{A \omega }{1- \omega }$ using $f(\omega) = 0$ $1 \leq (A + 1) \omega$ (with sign of $1 - \omega$ justified)</p> <p>$A + 1 \geq 1 \geq \omega$</p>	<p>M1</p> <p>A1 (AG)</p> <p>B1 (AG)</p> <p>(3 marks)</p>
	(c)	<p>If $\omega > 1$, $0 = \omega^n f\left(\frac{1}{\omega}\right)$ $= 1 + a_{n-1}\omega + \dots + a_1\omega^{n-1} + \omega^n$ Inequalities continue to hold since $a_i \leq A$</p> <p>If $\omega = 1$, then $1 + A \geq 1 \geq \frac{1}{1+A}$ since $A > 0$</p>	<p>M1</p> <p>E1</p> <p>E1</p> <p>(3 marks)</p>
	(ii)	<p>$f(x) = x^5 - x^4 - \frac{100}{135}x^3 - \frac{91}{135}x^2 - \frac{126}{135}x + 1$ Use $A = 1$. Integer roots with $\frac{1}{2} \leq \omega \leq 2$ could only be ± 1 or ± 2</p> <p>$f(\pm 2) \neq 0$ because numerator is odd (or any valid justification) $f(1) = -\frac{182}{135} \neq 0$ $f(1) = 0$</p> <p>$x = 1$ is the only integer root.</p>	<p>B1</p> <p>M1</p> <p>M1</p> <p>E1</p> <p>A1</p> <p>A1</p> <p>(6 marks)</p>

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4	(i)	$\sin \frac{\pi}{9} \cos \frac{\pi}{9} \cos \frac{2\pi}{9} \cos \frac{4\pi}{9}$ $= \frac{1}{2} \sin \frac{2\pi}{9} \cos \frac{2\pi}{9} \cos \frac{4\pi}{9}$ $= \frac{1}{8} \sin \frac{8\pi}{9}$ $= \frac{1}{8} \sin \frac{\pi}{9} \text{ (use of } \sin(\pi - x) = \sin(x))$ $\cos \frac{\pi}{9} \cos \frac{2\pi}{9} \cos \frac{4\pi}{9} = \frac{1}{8}$	B1 M1 M1 A1 (4 marks)
	(ii)	$\sin \left(\frac{x}{2^n} \right) \prod_{k=0}^n \cos \left(\frac{x}{2^k} \right)$ $= \frac{1}{2} \sin \left(\frac{x}{2^{n-1}} \right) \prod_{k=0}^{n-1} \cos \left(\frac{x}{2^k} \right)$ $= \dots \text{ (convincing use of induction or repeated application)}$ $= \frac{\sin(2x)}{2^{n+1}} \text{ (induction end point correct)}$ $\prod_{k=0}^n \cos \left(\frac{x}{2^k} \right) = \frac{\sin(2x)}{2^{n+1} \sin \left(\frac{x}{2^n} \right)}$ $\sum_{k=0}^n \log \left(\cos \left(\frac{x}{2^k} \right) \right) = \log(\sin(2x)) - \log \left(\sin \left(\frac{x}{2^n} \right) \right) - \log(2^{n+1})$ $\sum_{k=0}^n \frac{1}{2^k} \tan \left(\frac{x}{2^k} \right) = -2 \cot(2x) + \frac{1}{2^n} \cot \left(\frac{x}{2^n} \right)$ (justified with differentiation)	B1 M1 E1 A1 M1 (diff) M1 (division) A1 (AG) (7 marks)
	(iii)	B1 – switch to product starting at 0 M1 – set up as limiting case of product to n M1 – apply small angle for sin A1 – correct answer $\prod_{k=1}^n \cos \left(\frac{x}{2^k} \right) = \frac{\sin(2x)}{2^{n+1} \sin \left(\frac{x}{2^n} \right) \cos(x)}$ $= \frac{2 \sin(x)}{2^{n+1} \sin \left(\frac{x}{2^n} \right)}$ $\sim \frac{\sin(x)}{2^n \times \left(\frac{x}{2^n} \right)}$ $= \frac{\sin(x)}{x}$	M1 M1 M1 A1 (AG)

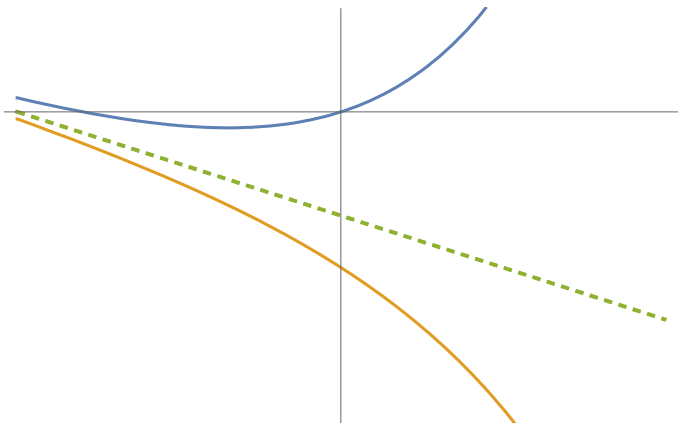
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		$\sum_{j=2}^n \frac{1}{2^{j-2}} \tan\left(\frac{\pi}{2^j}\right)$	M1
		$= \sum_{k=0}^n \frac{1}{2^k} \tan\left(\frac{\pi/4}{2^k}\right)$	M1
		$= \lim_{n \rightarrow \infty} \left(\frac{1}{2^n} \cot\left(\frac{\pi/4}{2^n}\right) - 2 \cot\left(\frac{\pi}{2}\right) \right)$	M1
		$= \lim_{n \rightarrow \infty} \left(\frac{1}{2^n \tan\left(\frac{\pi/4}{2^n}\right)} \right)$	M1
		$= \lim_{n \rightarrow \infty} \left(\frac{1}{2^n \left(\frac{\pi/4}{2^n}\right)} \right)$	M1
		$= \frac{4}{\pi}$	A1
			(9 marks)

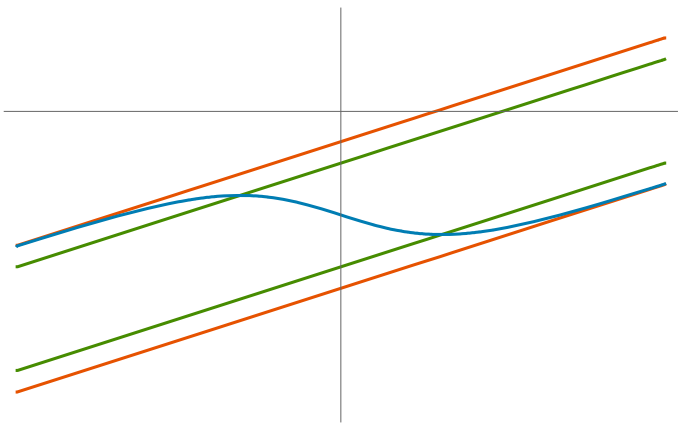
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5	(i)	<p>Constant iff $a = f(a)$ $\Leftrightarrow a = p + (a - p)a$ $\Leftrightarrow 0 = (a - p)(a - 1)$ $\Leftrightarrow a = p$ or $a = 1$.</p> <p>Period 2 $\Leftrightarrow a = f(f(a))$ $\Leftrightarrow 0 = (a - p)(-1 + 2ap - pa^2 + a^3)$ (factorisation) $\Leftrightarrow 0 = (a - p)(a - 1)(a^2 + (1 - p)a + 1)$</p> <p>If $a = p$ or $a = 1$, then sequence is constant. The quadratic has solutions when $(p - 1)^2 \geq 4$. If $(p - 1)^2 > 4$, i.e. $p > 3$ or $p < -1$, the solutions are distinct. They are not both 1, p since the sum of the roots is $p - 1 \neq p + 1$ So for $p > 3$ or $p < -1$, one of the roots of the quadratic gives a sequence of period 2.</p> <p>If $p = 3$, $a = 1$ so not period 2. If $p = -1$, $a = -1 = p$ so not period 2.</p>	<p>M1 M1 A1</p> <p>M1 M1 A1</p> <p>B1 M1</p> <p>E1 E1 (AG)</p> <p>B1 B1</p> <p>(12 marks)</p>
	(ii)	<p>No value of a for which the sequence is constant $\Leftrightarrow f(a) = a$ has no solution $\Leftrightarrow f(x) > x$ or $f(x) < x$ for all x</p> <p>But $f(x) > x$ for large x. So cannot have $f(x) < x$ for all x.</p> <p>If no value of a for which sequence constant, then $f(x) > x$ for all x So $f(f(x)) > f(x) > x$ for all x And hence no solution to $f(f(a)) = a$.</p> <p>Setting $p = q$, gives (i). Then if $-1 \leq p \leq 3$, there is no period 2 sequence but a constant sequence exists.</p>	<p>E1 (\rightarrow) E1 (\leftarrow)</p> <p>E1</p> <p>E1 E1 E1</p> <p>E1 E1</p> <p>(8 marks)</p>

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6	(i)	<p>If $y = mx + c$, Then the differential equation becomes $m = mx + c + x + 1$ $m = -1, c = -2$ $y = -x - 2$</p> <p>$\frac{dy}{dx} = 0 \Rightarrow y + x + 1 = 0 \Rightarrow y = -x - 1$</p> <p>$y = y_3(x)$ cannot cross the line $y = -x - 2$. So if $y_3(0) < -2$, it cannot reach the line $y = -x - 1$ and hence has no stationary points.</p> <p>At a stationary point, $\frac{d^2y}{dx^2} = \frac{dy}{dx} + 1 = y + x + 2 = 1 > 0$ so minimum</p> <p>$\frac{dY}{dx} = Y + 2$ $\log(Y + 2) = x + c$ $Y = -2 + Ae^x$ $y = -x - 2 + Ae^x$</p> <p>$y(0) = 0 \Rightarrow A = 2$ $y(0) = -3 \Rightarrow A = -1$ (attempt at both)</p> <p>So $y = -x - 2 + 2e^x$ So $y = -x - 2 - e^x$ (both)</p>  <p>Curves tending to asymptote to the left Curve above line through origin tending to ∞ Curve below line tending to $-\infty$</p>	<p>M1 A1 E1 (AG) E1 E1 M1 M1 A1 M1 G1 G1 G1</p> <p>(12 marks)</p>
	(ii)	<p>If $y = mx + c$, Then the differential equation becomes $m = (mx + c)^2 + 4(mx + c) + x^2 - 4x - 2x(mx + c) + 3$ $0 = (m^2 - 2x + 1)x^2 + (2mc + 4m - 4 - 2c) + c^2 + 4c + 3 - m$</p> <p>From x^2: $m = 1$ From x: $2mc + 4m - 4 - 2c = 2c + 4 - 4 - 2c = 0$</p>	

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		<p>From 1: $c^2 + 4c + 2 = 0 \Rightarrow c = -2 \pm \sqrt{2}$ Any of these equations Correct values of m and c</p> <p>Solutions: $y = x - 2 \pm \sqrt{2}$</p> <p>$\frac{dy}{dx} = (y - x)^2 + 4(y - x) + 3$ (writing as a function of $y - x$) $= (y - x + 3)(y - x + 1)$</p> <p>Stationary pts: $y = x - 1$ or $y = x - 3$</p> <p>Between these lines the gradient is negative. (Correctly justified)</p> <p>So stationary points on $y = x - 1$ are maxima and stationary on $y = x - 3$ are minima.</p>  <p>Curve does not intersect other solutions Curve has stationary points on correct lines</p>	<p>M1 A1</p> <p>M1</p> <p>A1</p> <p>A1</p> <p>A1</p> <p>G1 G1</p> <p>(8 marks)</p>
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7	(i)	$\mathbf{a} \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c}) = 0$ $\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} = -1$ <u>and cyclic permutations</u> $\mathbf{a} \cdot \mathbf{b} = -\frac{1}{2}$ legitimately obtained $\cos \theta = -\frac{1}{2}$ where θ is the angle between \mathbf{a} and \mathbf{b} $\theta = 120^\circ$ Similarly, the angle between \mathbf{a} and \mathbf{b} is 120° . Justification of equilateral triangle by sketch or otherwise ABC is equilateral	M1 M1 A1 M1 A1 M1 M1 A1 (8 marks)
	(ii)	$\mathbf{a}_1 \cdot \mathbf{a}_2 + \mathbf{a}_1 \cdot \mathbf{a}_3 + \mathbf{a}_1 \cdot \mathbf{a}_4 = -1$ <u>and cyclic permutations</u> Linear combination of these equations $\mathbf{a}_1 \cdot \mathbf{a}_2 = \mathbf{a}_3 \cdot \mathbf{a}_4$ (legitimately obtained)	M1 M1 A1 (AG) (3 marks)
	(a)	Angles $\angle A_1 O A_2 = \angle A_3 O A_4$ By symmetry, $\angle A_2 O A_3 = \angle A_4 O A_1$ The \mathbf{a}_i are distinct and unit length so no angles are zero (accept justification by sketch) $A_1 A_2 A_3 A_4$ is a rectangle	M1 M1 A1 (3 marks)
	(b)	$(A_1 A_2)^2 = (\mathbf{a}_1 - \mathbf{a}_2)^2$ $= \mathbf{a}_1^2 + \mathbf{a}_2^2 - 2\mathbf{a}_1 \cdot \mathbf{a}_2$ $= 2 - 2\mathbf{a}_1 \cdot \mathbf{a}_2$ By symmetry, $\mathbf{a}_1 \cdot \mathbf{a}_2 = \mathbf{a}_1 \cdot \mathbf{a}_3 = \mathbf{a}_1 \cdot \mathbf{a}_4$ So $\mathbf{a}_1 \cdot \mathbf{a}_2 = -\frac{1}{3}$ So $(A_1 A_2)^2 = \frac{8}{3}$ $A_1 A_2 = \frac{2\sqrt{2}}{\sqrt{3}}$	M1 M1 M1 A1 M1 A1 (6 marks)

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8	(i)	$f(\mathbf{M}) = f(\mathbf{MI}) = f(\mathbf{M})f(\mathbf{I})$ $\Rightarrow f(\mathbf{I}) = 1 \text{ since } f(\mathbf{M}) \neq 0$	M1 A1 (AG) (2 marks)
	(ii)	$f(\mathbf{J})^2 = f(\mathbf{J}^2)$ $= f(\mathbf{I}) = 1$ $\Rightarrow f(\mathbf{J}) = -1 \text{ since } f(\mathbf{J}) \neq 1$ $f\left(\begin{pmatrix} c & d \\ a & b \end{pmatrix}\right) = f\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$ $= -f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \text{ legitimately obtained}$ $f\left(\begin{pmatrix} d & c \\ b & a \end{pmatrix}\right) = f\left(\begin{pmatrix} c & d \\ a & b \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$ $= -f\left(\begin{pmatrix} c & d \\ a & b \end{pmatrix}\right) \text{ legitimately obtained}$	M1 M1 A1 M1 A1 (AG) M1 A1 (AG) (7 marks)
	(iii)	<u>Using first equality in previous part</u> (or otherwise correctly justified) $f\left(\begin{pmatrix} a & b \\ a & b \end{pmatrix}\right) = -f\left(\begin{pmatrix} a & b \\ a & b \end{pmatrix}\right)$ $f\left(\begin{pmatrix} a & b \\ a & b \end{pmatrix}\right) = 0$ $f\left(\begin{pmatrix} a & b \\ ka & kb \end{pmatrix}\right) = f\left(\mathbf{K} \begin{pmatrix} a & b \\ a & b \end{pmatrix}\right)$ $= 0$	M1 M1 M1 A1 (AG) (4 marks)
	(iv)	$\mathbf{K}^{-1}\mathbf{PK} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ $f(\mathbf{K})f(\mathbf{K}^{-1}) = f(\mathbf{I}) = 1 \Rightarrow f(\mathbf{K}^{-1}) = f(\mathbf{K})^{-1}$ $f(\mathbf{K}^{-1}\mathbf{PK}) = f(\mathbf{K}^{-1})f(\mathbf{PK}) \text{ (must use two stages)}$ $= f(\mathbf{K}^{-1})f(\mathbf{P})f(\mathbf{K})$ $= f(\mathbf{P})$ $\mathbf{P}^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ $f(\mathbf{P}^2) = f(\mathbf{P}) \Rightarrow f(\mathbf{P}) = 0 \text{ or } 1$ $\mathbf{P}^{-1} \text{ exists so } f(\mathbf{P})f(\mathbf{P}^{-1}) = 1 \Rightarrow f(\mathbf{P}) \neq 0$	B1 M1 M1 A1 (AG)? B1 M1 A1 (7 marks)

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9	(i)	$\mathbf{r} = \begin{pmatrix} ut \cos \alpha \\ ut \sin \alpha - \frac{1}{2}gt^2 \end{pmatrix}$ $\mathbf{r}^2 = u^2t^2 \cos^2 \alpha + u^2t^2 \sin^2 \alpha - ugt^3 \sin \alpha + \frac{1}{4}g^2t^4$ $= u^2t^2 - ugt^3 \sin \alpha + \frac{1}{4}g^2t^4$ $\frac{d}{dt}(\mathbf{r}^2) = 2u^2t - 3ugt^2 \sin \alpha + g^2t^3$ $= t(2u^2 - 3ugt \sin \alpha + g^2t^2)$ $= t(2u^2 - \frac{9}{4}u^2 \sin^2 \alpha + (gt - \frac{3}{2}u \sin \alpha)^2)$ <p>If $\sin \alpha < \frac{2\sqrt{2}}{3}$, then $2u^2 - \frac{9}{4}u^2 \sin^2 \alpha > 0$ and distance is always increasing.</p> <p>If $\sin \alpha > \frac{2\sqrt{2}}{3}$, then distance is decreasing at $t = \frac{3u}{2g} \sin \alpha$ Landing occurs at $t = \frac{2u}{g} \sin \alpha$, which is later (Or imagine falls through ground. Distance increasing while underground, so any decrease must be above ground)</p>	<p>M1</p> <p>M1 A1</p> <p>M1 A1 M1 M1</p> <p>M1 A1 (AG)</p> <p>M1 E1</p> <p>(11 marks)</p>
	(ii)	$\mathbf{r} = \begin{pmatrix} ut \cos \alpha + vt \\ ut \sin \alpha - \frac{1}{2}gt^2 \end{pmatrix}$ $PQ^2 = (u \cos \alpha + v)^2t^2 + u^2t^2 \sin^2 \alpha - ugt^3 \sin \alpha + \frac{1}{4}g^2t^4$ $\frac{d}{dt}(PQ^2) = 2t(u \cos \alpha + v)^2 + 2u^2t \sin^2 \alpha + 2tu^2 \sin^2 \alpha - 3ugt^2 \sin \alpha + g^2t^3$ $= t \left(2(u \cos \alpha + v)^2 + 2u^2 \sin^2 \alpha - \frac{9}{4}u^2 \sin^2 \alpha + \left(gt - \frac{3}{2}u \sin \alpha \right)^2 \right)$ $= t \left(2(u \cos \alpha + v)^2 - \frac{1}{4}u^2 \sin^2 \alpha + \left(gt - \frac{3}{2}u \sin \alpha \right)^2 \right)$ <p>If $2\sqrt{2}v > (\sin \alpha - 2\sqrt{2} \cos \alpha)u$, then $8(u \cos \alpha + v)^2 > u^2 \sin^2 \alpha$ So PQ is increasing for all t.</p>	<p>B1</p> <p>M1 A1</p> <p>M1</p> <p>M1</p> <p>M1 A1</p> <p>M1 A1 (AG)</p> <p>(9 marks)</p>

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10	(i)	<p>Correct diagram</p> <p>Moments about A: $Wa \cos \theta (1 + 2k) = 2aT \sin 2\theta$ If $2k + 1 > 4 \sin \theta$ then $2T \sin 2\theta > W \cos \theta (4 \sin \theta) = 2W \sin 2\theta$ Since $\sin 2\theta > 0$, $T > W$ and so the string will break.</p>	<p>B2</p> <p>M1</p> <p>M1</p> <p>A1</p> <p>A1 (AG)</p> <p>(6 marks)</p>
	(ii)	<p>Resolving forces vertically: $R = ((k + 1)W - T \sin \theta)$ Resolving horizontally, ring will slip if: $T \cos \theta > \mu((k + 1)W - T \sin \theta)$ (= max value for friction) Moments about A: $W(2k + 1) = 4T \sin \theta$ $\mu((k + 1)W - T \sin \theta) = \mu \left(\frac{4(k+1)}{2k+1} - 1 \right) T \sin \theta$ $\mu \left(\frac{2k+3}{2k+1} \right) T \sin \theta$</p> <p>If $2k + 1 > (2k + 3)\mu \tan \theta$, then $\mu \left(\frac{2k+3}{2k+1} \right) \sin \theta < \cos \theta$ So the ring will slip.</p>	<p>M1</p> <p>M1</p> <p>M1</p> <p>A1</p> <p>M1</p> <p>A1 (AG)</p> <p>(6 marks)</p>
	(iii)	<p>Attempt to solve breaking inequality for k Breaks at $k = \frac{4 \sin \theta - 1}{2}$</p> <p>Attempt to solve slipping inequality for k Slips at $k = \frac{3\mu \tan \theta - 1}{2(1 - \mu \tan \theta)}$</p> <p>If ring slips before it breaks: $\frac{3\mu \tan \theta - 1}{2(1 - \mu \tan \theta)} < \frac{4 \sin \theta - 1}{2}$ (for A1, do not allow $>$)</p> <p>Confirming that inequality is being multiplied by a positive quantity.</p> <p>$3\mu \tan \theta - 1 < 4 \sin \theta - 1$ $\mu < \frac{2 \cos \theta}{2 \sin \theta + 1}$</p>	<p>M1</p> <p>A1</p> <p>B1</p> <p>M1 A1</p> <p>E1</p> <p>M1 A1</p> <p>(AG)</p> <p>(8 marks)</p>

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11	(i)	<p>In both cases, award the M mark if all possible values of n_2 for at least 3 values of n_1 are identified.</p> <p>$n_3 = 9$ $n_1 = 1; n_2$ has no options $n_1 = 2; n_2 = 8$ $n_1 = 3; n_2 = 8, 7$ $n_1 = 4; n_2 = 8, 7, 6$ $n_1 = 5; n_2 = 8, 7, 6$ $n_1 = 6; n_2 = 8, 7$ $n_1 = 7; n_2 = 8$ $n_1 = 8; n_2$ has no options</p> <p>Total = $(1 + 2 + 3) \times 2 = 12$</p> <p>$n_3 = 10$ $n_1 = 1; n_2$ has no options $n_1 = 2; n_2 = 9$ $n_1 = 3; n_2 = 9, 8$ $n_1 = 4; n_2 = 9, 8, 7$ $n_1 = 5; n_2 = 9, 8, 7, 6$ $n_1 = 6; n_2 = 9, 8, 7$ $n_1 = 7; n_2 = 9, 8$ $n_1 = 8; n_2 = 9$ $n_1 = 0; n_2$ has no options</p> <p>Total = $(1 + 2 + 3 + 4) \times 2 - 4 = 16$</p> <p>$n_3 = 2n + 1$ Total ways = $(1 + \dots + (n - 1)) \times 2$ (method mark may be implicit) = $(n - 1)n$</p> <p>$n_3 = 2n$ Total ways = $(1 + \dots + (n - 1)) \times 2 - (n - 1)$ (method mark may be implicit) = $(n - 1)^2$</p>	<p>M1</p> <p>M1</p> <p>A1 (both totals correct)</p> <p>M1 A1</p> <p>M1 A1</p> <p>(7 marks)</p>
	(ii)	<p>Total number of pairs is $\binom{N-1}{2} = \frac{1}{2}(N-1)(N-2)$</p> <p>Justification for using first part of question</p> <p>$N = 2n + 1$ Prob = $\frac{(n-1)n}{\frac{1}{2}(2n)(2n-1)} = \frac{n-1}{2n-1}$</p> <p>$N = 2n$ Prob = $\frac{(n-1)^2}{\frac{1}{2}(2n-1)(2n-2)} = \frac{n-1}{2n-1}$</p>	<p>M1</p> <p>B1</p> <p>A1 (AG)</p> <p>A1</p> <p>(4 marks)</p>

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	(iii)	$\text{Prob} = \sum_{n=1}^M \frac{n-1}{2n-1} \times \mathbb{P}(\text{largest rod is } 2n+1) + \sum_{n=1}^M \frac{n-1}{2n-1}$ $\times \mathbb{P}(\text{largest rod is } 2n)$ $= \sum_{n=1}^M \frac{n-1}{2n-1} \left(\frac{\binom{2n}{2}}{\binom{2M+1}{3}} + \frac{\binom{2n-1}{2}}{\binom{2M+1}{3}} \right)$ $= \frac{6}{(2M+1)(2M)(2M-1)}$ $\cdot \frac{1}{2} \sum_{n=1}^M \frac{n-1}{2n-1} (2n(2n-1) + (2n-1)(2n-2))$ <p>(Use of formula for binomial coefficients with factorials cancelled)</p> $= \frac{3}{M(2M+1)(2M-1)} \sum_{n=1}^M (n-1)(2n-1)$ <p>Use of $\sum_1^K k^2 = \frac{1}{6}K(K+1)(2K+1)$ to simplify above</p> $= \frac{3}{M(2M+1)(2M-1)} \left(\frac{1}{3}M(M+1)(2M+1) - 3 \times \frac{1}{2}M(M+1) + M \right)$ $= \frac{1}{2(2M+1)(2M-1)} (4M^2 - 3M - 1)$ $= \frac{(4M+1)(M-1)}{2(2M+1)(2M-1)}$	<p>M1 A1 (ft)</p> <p>M1 A1</p> <p>M1</p> <p>M1</p> <p>M1</p> <p>M1 A1</p> <p>(9 marks)</p>
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12	(i)	$\mu = \int_0^1 nx^n dx = \frac{n}{n+1}$ $\mathbb{E}(X^2) = \int_0^1 nx^{n+1} dx = \frac{n}{n+2}$ $\sigma^2 = \frac{n}{n+2} - \left(\frac{n}{n+1}\right)^2 = \frac{n}{(n+1)^2(n+2)}$	<p>M1 A1</p> <p>M1</p> <p>M1 A1 (AG)</p> <p>(5 marks)</p>
	(ii)	$\text{LQ} = \frac{1}{2}, \text{UQ} = \frac{\sqrt{3}}{2}$ $\text{IQR} = \frac{\sqrt{3}-1}{2}$ $2\sigma = \frac{\sqrt{2}}{3}$ <p>Squaring IQR and 2σ</p> <p>Comparing $\sqrt{3}$ with a rational number... ...by squaring both sides</p> <p>Argument correct</p>	<p>M1</p> <p>A1</p> <p>B1</p> <p>M1</p> <p>M1</p> <p>M1</p> <p>A1</p> <p>(7 marks)</p>
	(iii)	$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \dots$ $+ \frac{n(n-1)\dots(n-k+1)}{k!}x^k + \dots$ $\text{LQ} = \left(\frac{1}{4}\right)^{1/n} \text{ and Median} = \left(\frac{1}{2}\right)^{1/n}$ $\left(\frac{1}{\mu}\right)^n = \left(1 + \frac{1}{n}\right)^n > 1 + n\left(\frac{1}{n}\right) = 2$ <p>So $\mu < \left(\frac{1}{2}\right)^{1/n}$</p> $\left(\frac{1}{\mu}\right)^n = \left(1 + \frac{1}{n}\right)^n < 1 + n\left(\frac{1}{n}\right) + \frac{n^2}{2!}\left(\frac{1}{n}\right)^2 + \dots + \frac{n^k}{k!}\left(\frac{1}{n}\right)^k + \dots$ $< 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{k!} + \dots$ < 4 <p>So $\mu > \left(\frac{1}{4}\right)^{1/n}$</p>	<p>A1</p> <p>A1</p> <p>B1</p> <p>M1</p> <p>A1</p> <p>M1</p> <p>M1</p> <p>A1</p> <p>(8 marks)</p>